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Supersymmetric Wigner–Dunkl quantum mechanics

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ABSTRACT

Wigner–Dunkl quantum mechanics can be viewed as deformed quantum mechanics, where the commutator between canonical momentum and position operators contains in addition a reflection operator. In the present work we carry over the general concepts of supersymmetric (SUSY) quantum mechanics to the Wigner–Dunkl quantum formulation. We investigate the SUSY ground states, the SUSY transformations and introduce a generalized shape invariance allowing for explicit solutions. In this way, exact solutions for a certain class of SUSY potentials can be provided. As example the harmonic oscillator is considered. We also take a look into a singular oscillator model, which does not belong to the shape-invariance class. Nevertheless, exact solutions can be provided, showing the peculiarities of that model in the SUSY Wigner–Dunkl quantum mechanics. A modification of that singular model is presented which restores shape invariance

Introduction

Exactly solvable models in quantum mechanics are rare but of vital interest. Schrödinger [1] himself introduced, what nowadays is called the factorization method, to find exactly solvable quantum models. This method was analyzed in some depth by Infeld and Hull [2], who made an exhaustive classification of factorizable potentials. Famous example are the harmonic oscillator, Coulomb, Morse and Pöschl–Teller potentials, which found extensive use in nuclear, atomic and molecular physics. For more details see, for example, the book by Dong [3]. The factorization method, which in essence goes even further back to Darboux [4] and is also known under the name Darboux transformation, has seen an extensive revival with Witten's model of supersymmetric quantum mechanics (SUSYQM) [5]. The factorization condition, also called shape-invariance in SUSYQM, has triggered enormous efforts to find further exactly solvable, quasi-exactly solvable and conditionally exactly solvable models. For an overview, see the recent book by Junker [6].

Witten's one-dimensional model of SUSYQM consists of an essential isospectral pair of supersymmetric partner Hamiltonians $H_\pm$. That is, the strictly positive part of the spectrum of both Hamiltonians is identical. With the so-called SUSY charge operator A and its adjoint $A^\dagger$ both Hamiltonians factorize into $H_- = A^\dagger A$ and $H_+ = AA^\dagger$. Obviously, both have non-negative eigenvalue and the strictly positive eigenvalues must coincide. SUSY is said to be unbroken if at least one of these Hamiltonians has a zero eigenvalue.

The purpose of the present work is to extent the general concepts of Witten's model of SUSYQM to the Wigner–Dunkl quantum mechanics (WDQM). The WDQM goes back to Wigner's approach [7] where he introduced the reflection operator together with a parameter, say ν, in the commutation relation between the momentum and position operator. Shortly after that Yang [8] has found the coordinate representation, in which the momentum operator is represented by the Dunkl derivative [9] being a generalized first-order derivative. Up to now there have been much studies related to the Dunkl derivative [10–12]. The idea to combine both, WDQM and SUSYQM, is also not new and has been studied by various authors starting with Plyushchay [13–15] and more recently followed by others [16,17]. Here we like to reconsider the general concepts of supersymmetric WDQM such as the SUSY ground states, the SUSY transformations and the shape-invariance.

This paper is organized as follows. In the next section we recall some basics of WDQM. Section "Supersymmetric Wigner–Dunkl quantum mechanics" then starts with the standard SUSYQM, which then is generalized to the WDQM with the help of the Dunkl derivative. Some basic properties like the SUSY ground state and the SUSY transformations are discussed within this modified SUSYQM. Section "Generalized shape invariance" discusses a generalized shape invariance conditions, which

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may be used for exact solutions of SUSY WDQM. As an example, the harmonic oscillator is discussed explicitly in Section "The harmonic oscillator". Finally, a singular extension of this harmonic oscillator, being not shape invariant, is considered in Section "The singular harmonic oscillator". Due to its singular SUSY potential we find an additional double degeneracy of energy levels for that model. We also briefly discuss a modified version of the singular harmonic oscillator which exhibits a shape invariance. Some conclusions are drawn in Section "Discussion".

Preliminaries

In this section we briefly recall the definition and some basic properties of the Dunkl derivative and the associated Wigner–Dunkl modification of Schrödinger's equation. The Dunkl derivative, in essence, is a non-local generalization of the usual first-order derivative including the reflection operator R

$$D_x = \partial_x + \frac{\nu}{x}(1-R)\,. \tag{1}$$

Here $\nu > -1/2$ is a real parameter and R is the reflection (or parity or Klein) operator acting on an arbitrary function on the real line as follows $(Rf)(x) = f(-x)$. That is, the value of a function f at point x is mapped non-locally onto the value of f at point $-x$. Hence, R represents a reflection of f at the origin $x = 0$.

The generalized momentum operator $\hat{p}$ may thus be defined as

$$\hat{p} = -i\,D_x, \tag{2}$$

and was first introduced by Yang in 1951 when discussing a possible generalized quantization of the harmonic oscillator following Wigner's original discussion. Note that we will set $\hbar = 1$ throughout this paper.

One of the consequences of above generalization is a deformed Heisenberg relation

$$[\hat{x},\hat{p}] = i\,(1+2\nu R)\,. \tag{3}$$

Another one is, that the standard Hamiltonian given by $\hat{H} = \hat{p}^2/2\,m + V(x)$ leads us to the time-independent Dunkl–Schrödinger equation

$$\left(-\frac{1}{2m}D_x^2 + V(x)\right)\psi(x) = E\psi(x) \tag{4}$$

or more explicitly to

$$\left[-\frac{1}{2m}\left(\partial_x^2 + \frac{2\nu}{x}\partial_x - \frac{\nu}{x^2}(1-R)\right) + V(x)\right]\psi(x) = E\psi(x)\,. \tag{5}$$

Let us note here, that the Hilbert space is given by $\mathcal{H} = L^2(\mathbb{R}, \mathrm{d}x|x|^{2\nu})$. In other words, the scalar product on $\mathcal{H}$ is defined by

$$\langle\psi|\varphi\rangle = \int_{-\infty}^{\infty} \mathrm{d}x|x|^{2\nu}\,\psi^*(x)\varphi(x) \tag{6}$$

In particular, the Dunkl derivative is skew-Hermitian with respect to that measure,

$$\int_{-\infty}^{\infty} \mathrm{d}x|x|^{2\nu}\psi^*(x)D_x\psi(x) = \int_{-\infty}^{\infty} \mathrm{d}x|x|^{2\nu}(-D_x\psi(x))^*\psi(x), \tag{7}$$

and hence the momentum operator is Hermitian on $\mathcal{H}$. Obviously, for $\nu = 0$ we are back at the standard un-deformed quantum mechanics case.

Finally, let us point out that whenever the potential V is invariant under R, i.e. $[V, R] = 0$ implying $(RV)(x) = V(x)$, then R also commutes with the full Hamiltonian and represents a conserved quantity. In particular, the energy eigenstates of (5) are also eigenstates of R with either even or odd parity. A discussion of above properties as well as more details can be found in ref. [12].

Supersymmetric Wigner–Dunkl quantum mechanics

Supersymmetric quantum mechanics (SUSYQM) is dealing with a pair of Hamiltonians being expressed in terms of a supercharge operator A and its adjoint as follows

$$H_+ = AA^\dagger\,, \qquad H_- = A^\dagger A\,. \tag{8}$$

It is obvious that both have an identical spectrum away from zero as any eigenstate with a strictly positive eigenvalue of one Hamiltonian can be mapped into an eigenstate of its partner Hamiltonian by acting on it with operator A, respectively $A^\dagger$. Such SUSY transformations do not necessarily exist for states within the kernel of A and $A^\dagger$. A simple but instructive model is the one-dimensional Witten model of SUSYQM. Here the supercharge is given by a generalized annihilation operator $A = (\partial_x + W(x))/\sqrt{2m}$, where $m > 0$ is the mass of the quantum particle moving along the real x-axis. For a system where the SUSY potential W is an odd function, i.e. $(RW)(x) = W(-x) = -W(x)$, it is known that SUSY is unbroken as there exists a zero-energy eigenstate $\psi_0^-(x)$ for H_- given via $A\psi_0^-(x) = 0$.

The objective of this section is to generalize the standard SUSY approach of Witten's model to the Wigner–Dunkl quantum mechanics. That is, we are going to replace the usual derivative by the Dunkl derivative, which is slightly different to the original form introduced by Plyushchay [15] for the case of a linear SUSY potential. Hence, from now on, we set

$$A = \frac{1}{\sqrt{2m}}\left(D_x + W(x)\right) \tag{9}$$

and assume that the SUSY potential is an odd function of x. An odd SUSY potential was also considered in [17] however, with yet another variant of (9). A typical SUSY potential we are considering here would be of the form $W(x) \sim x^{2k+1}$ with $k = 0, 1, 2, 3, \ldots$. To be a bit more general we will only assume $W(-x) = -W(x)$ and $\lim_{x\to\pm\infty} W(x) = \pm\infty$. As a consequence the SUSY potential as well as A and its adjoint $A^\dagger$ anti-commute with the parity operator, i.e.

$$\{W, R\} = 0 \qquad \text{and} \qquad \{A, R\} = 0 = \{A^\dagger, R\}. \tag{10}$$

Having set up the stage, let us take a closer look at the SUSY partner Hamiltonians (8). An explicit calculation shows that

$$\begin{aligned} H_\pm &= -\frac{1}{2m}D_x^2 + \frac{1}{2m}W^2(x) \pm \frac{1}{2m}[D_x, W(x)] \\ &= -\frac{1}{2m}\left[\partial_x^2 + \frac{2\nu}{x}\partial_x - \frac{\nu}{x^2}(1-R)\right] \\ &\quad + \frac{1}{2m}W^2(x) \pm \frac{1}{2m}\left(W'(x) + \frac{2\nu}{x}W(x)R\right). \end{aligned} \tag{11}$$

Obviously, $[H_\pm, R] = 0$ and therefore, the energy eigenstates of $H_\pm$ may also be eigenstates of the parity operator with eigenvalue $+1$ and -1 for an even and odd energy eigenfunction, respectively. Let us recall here that these eigenstates live in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}, \mathrm{d}x|x|^{2\nu})$ equipped with scalar product (6). We also note here that for $\nu = 0$ the above partner Hamiltonians (11) are those of the standard Witten model of SUSYQM. Finally, by comparing (8) with the Wigner–Dunkl Hamiltonian (5) we may introduce the partner potentials

$$V_\pm(x) = \frac{1}{2m}W^2(x) \pm \frac{1}{2m}\left(W'(x) + \frac{2\nu}{x}W(x)R\right) \tag{12}$$

resulting in the form

$$H_\pm = -\frac{1}{2m}D_x^2 + V_\pm(x) \geq 0. \tag{13}$$

The SUSY ground state

For unbroken SUSY there necessarily exists a zero energy eigenstate which either belongs to H_- or H_+ [5,6]. That is, this ground state either is in the kernel of A or $A^\dagger$. We will see later that even both kernels might contain a SUSY ground state. Let us discuss the various scenarios.

First, we assume that the SUSY ground state belongs to H_-, denoted by ψ_0^-. Hence, it obeys the relation

$$A\psi_0^- = 0 \qquad \Longleftrightarrow \qquad \left(\partial_x + \frac{\nu}{x}(1-R) + W(x)\right)\psi_0^-(x) = 0, \tag{14}$$

which is easily integrated. For this we need to differentiate between the two cases where this ground state is either an even or odd function, which we indicate with an additional subindex e or o, respectively. For the even case, $(1-R)\psi_{0e}^- = 0$, we immediate find

$$\psi_{0e}^-(x) = N_- \exp\left\{-\int^x \mathrm{d}z\, W(z)\right\} \tag{15}$$

which is similar in form to the SUSY ground state of standard SUSYQM. The normalization constant is given by

$$\frac{1}{N_-^2} = \int_{-\infty}^{\infty} \mathrm{d}x\, |x|^{2\nu} \exp\left\{-2\int^x \mathrm{d}z\, W(z)\right\}, \tag{16}$$

which obviously exits for the class of SUSY potentials we considering here. However, in the case of an odd ground state $(1-R)\psi_{0o}^- = 2\psi_{0o}^-$ above relation (14) results in

$$\psi_{0o}^-(x) \sim \mathrm{sgn}\,(x)|x|^{-2\nu} \exp\left\{-\int^x \mathrm{d}z\, W(z)\right\} \tag{17}$$

with $\mathrm{sgn}\,(x) = x/|x|$ being the usual sign function. Due to the singular term $|x|^{-4\nu}$, appearing in the normalization integral, this function is not integrable at $x = 0$ with respect to the scalar product (6) for $\nu > -1/2$. Hence, no odd ground state exists for the class of models under consideration. However, as we will show in Section “The singular harmonic oscillator”, when allowing for a singular SUSY potential which contains a $1/x$-term this might result in a normalizable wave function (17) and thus an odd ground state might exist, too.

Let us now consider the second case where we assume that the zero-energy ground state belongs to H_+. In that case, this ground state ψ_0^+ is defined via

$$A^\dagger\psi_0^+ = 0 \qquad \Longleftrightarrow \qquad \left(\partial_x - \frac{\nu}{x}(1-R) - W(x)\right)\psi_0^+(x) = 0. \tag{18}$$

For ψ_0^+ being an even function this will result in a form

$$\psi_{0e}^+(x) \sim \exp\left\{+\int^x \mathrm{d}z\, W(z)\right\}, \tag{19}$$

which clearly is not normalizable for the class of SUSY potentials we are considering. Similarly, assuming an odd ground state $(1-R)\psi_{0o}^+ = 2\psi_{0o}^+$ would lead us to a solution of the form

$$\psi_{0o}^-(x) \sim \mathrm{sgn}\,(x)|x|^{2\nu} \exp\left\{+\int^x \mathrm{d}z\, W(z)\right\} \tag{20}$$

and again the diverging exponent would not allow for a normalization.

In summarizing this section we can conclude, that for the class of odd SUSY potentials with $\lim_{x\to\pm\infty} W(x) = \pm\infty$, SUSY is unbroken with a zero energy eigenstate given by (15) belonging to H_-. If we, in addition, allow for a singular SUSY potential containing a term like λ/x the ground state (17) is normalizable if λ is chosen in a proper range. We will set aside this singular case for a while, but come back to it in Section “The singular harmonic oscillator”, where an explicit example is discussed.

SUSY transformations and parity of energy eigenstates

As in standard SUSYQM we may relate the positive-energy eigenstates of the partner Hamiltonians via so-called SUSY transformations. From definition (8) immediately follows the relation

$$H_+A = AH_- \qquad \text{and} \qquad A^\dagger H_+ = H_-A^\dagger. \tag{21}$$

Furthermore, denoting the eigenstates of $H_\pm$ by $\psi_E^\pm$, that is, for $E > 0$ we set

$$H_\pm\psi_E^\pm = E\psi_E^\pm, \tag{22}$$

then we have the obvious SUSY transformations

$$\psi_E^+ = \frac{1}{\sqrt{E}}\,A\psi_E^- \qquad \text{and} \qquad \psi_E^- = \frac{1}{\sqrt{E}}\,A^\dagger\psi_E^+. \tag{23}$$

These relations imply that the strictly positive part of the spectrum of H_+ is identical to the strictly positive part of the spectrum of H_-

$$\mathrm{spec}\, H_+\backslash\{0\} = \mathrm{spec}\, H_-\backslash\{0\}. \tag{24}$$

In other words, the SUSY partner Hamiltonians are essentially iso-spectral as in usual SUSYQM [6].

In addition, as here we are dealing with an odd SUSY potential, these energy eigenstates are also eigenstates of the parity operator R. If we put

$$R\psi_E^\pm = r^\pm(E)\psi_E^\pm, \qquad \text{with} \qquad r^\pm(E) \in \{-1,+1\} \tag{25}$$

and recall that A and $A^\dagger$ anticommute with R, we conclude the relation

$$r^+(E) = -r^-(E), \qquad \forall\, E > 0. \tag{26}$$

Let us assume that H_- has a pure discrete spectrum, say $\{E_0, E_1, E_2, \ldots\}$ with $E_0 = 0$ and $E_{n+1} > E_n$, then it is obvious that the parity of its eigenstates is given by

$$r^-(E_n) = (-1)^n \tag{27}$$

starting with $n = 0$ for ground state (15).

Generalized shape invariance

The above discussion also suggests to take a close look at the shape-invariance condition known from standard SUSYQM [18]. Let us assume we are given a set of SUSY potentials $\{W(a_s,x)\}$, where $s = 0,1,2,\ldots$ and a_s stands for a fixed set of parameters. All elements of that set should obey previously made assumptions, that is, they should be of odd parity resulting in unbroken SUSY. As the Dunkl derivative (1) contains an additional R-term it is reasonable to propose a generalized shape-invariance condition as follows

$$V_+(a_{s-1},x) = V_-(a_s,x) + F(a_s) + G(a_s)\,\nu R. \tag{28}$$

That is, the partner potential V_+ can be expressed in terms of V_- with the help of a new set of parameters plus a constant term represented by $F(a_s)$ and another constant term being linear in R. The presence of this last R-dependent term is in fact the generalization of the standard shape-invariance condition [6].

The family of SUSY potentials results in an associated family of SUSY partner Hamiltonians

$$H_-(a_s) = A^\dagger(a_s)A(a_s), \quad H_+(a_s) = A(a_s)A^\dagger(a_s) \tag{29}$$

with

$$A(a_s) = \frac{1}{\sqrt{2m}}(D_x + W(a_s,x)), \tag{30}$$

which obey, due to condition (28), in addition the relation

$$H_+(a_{s-1}) = H_-(a_s) + F(a_s) + G(a_s)\,\nu R. \tag{31}$$

Hence we conclude

$$H_-(a_{s-1})A^\dagger(a_{s-1}) = A^\dagger(a_{s-1})H_+(a_{s-1}) = A^\dagger(a_{s-1}) \times \left(H_-(a_s) + F(a_s) + G(a_s)\,\nu R\right). \tag{32}$$

Let us now assume that we have an eigenstate of $H_-(a_s)$ with eigenvalue E_n,

$$H_-(a_s)\psi_{E_n}^-(a_s) = E_n\psi_{E_n}^-(a_s), \tag{33}$$

then, by applying (32) on that state, we may conclude that

$$\psi_{E_{n+1}}^-(a_{s-1}) = \frac{1}{\sqrt{E_{n+1}}}\,A^\dagger(a_{s-1})\psi_{E_n}^-(a_s) \tag{34}$$

is an eigenstate of $H_-(a_{s-1})$ with eigenvalue E_{n+1} given by

$$E_{n+1} = E_n + F(a_s) + \nu G(a_s) r^-(E_n)\,. \tag{35}$$

Hence, starting with the ground state of $H_-(a_n)$

$$\psi_0^-(a_n) = N_-(a_n) \exp\left\{ - \int^x \mathrm{d}z\, W(a_n, z) \right\}\,, \tag{36}$$

we may obtain the eigenvalues and eigenstates of

$$H_-(a_0) = -\frac{1}{2m} D_x^2 + V_-(a_0, x) \tag{37}$$

as follows.

$$\begin{aligned} E_n &= \sum_{s=1}^{n} \Big(F(a_s) + (-1)^{n-s} \nu G(a_s) \Big)\,, \\ \psi_n^-(a_0) &= \prod_{s=1}^{n} \left(\frac{A^\dagger(a_s)}{\sqrt{E_s}} \right) \psi_0^-(a_n)\,. \end{aligned} \tag{38}$$

Let us note that for $\nu = 0$ this result is identical to that found in standard SUSYQM. The difference in form is only the second term in above sum. Note that there we made use of the relation (27).

To conclude this section we like to mention that the generalized shape-invariance (28) imposes rather strict conditions on the family of SUSY potentials, which are

$$\begin{aligned} &W^2(a_{s-1}, x) + W'(a_{s-1}, x) = W^2(a_s, x) - W'(a_s, x) + 2\, mF(a_s)\,, \\ &W(a_{s-1}, x) = -W(a_s, x) + mG(a_s)\, x\,. \end{aligned} \tag{39}$$

These equations are obtained by separating the R-dependent from the R-independent terms in (28). In below section we will present an example of such a shape-invariant family of SUSY potentials.

The harmonic oscillator

As an ansatz let us consider the family of linear SUSY potentials $W(a_s, x) = a_s x$ with $a_s > 0$. Inserting this ansatz into the first equation of (39) leads us to conclude that all a_s must be identical and hence we set $a_s = m\omega$ with $\omega > 0$. This leads us to

$$F(a_s) = \omega \qquad \text{and} \qquad G(a_s) = 2\omega\,. \tag{40}$$

The corresponding SUSY partner Hamiltonians are given by

$$H_\pm = H_{HO} \pm \frac{\omega}{2}(1 + 2\nu R)\,, \tag{41}$$

where

$$H_{HO} = -\frac{1}{2m} D_x^2 + \frac{m}{2}\omega^2 x^2 \tag{42}$$

is the harmonic oscillator Hamiltonian of the Wigner–Dunkl quantum mechanics. Let us note here, that this system or variants of it have been studied in the literature since the early 1990s. See, for example, [12–15,19].

The energy eigenvalues of (41) are immediately obtained via (38) using the parameters (40)

$$E_n = \omega \sum_{s=1}^{n} \Big(1 + 2\nu(-1)^{n-s} \Big) = \omega[n]_\nu\,, \tag{43}$$

where we have introduced the Dunkl number

$$[n]_\nu = n + \nu(1 - (-1)^n)\,. \tag{44}$$

Note that in (43) we allow for $n = 0, 1, 2, 3, \ldots$ to represent the spectrum of H_-, whereas $n = 1, 2, 3, \ldots$ will enumerated that for H_+, respectively. As $H_{HO} = (H_+ + H_-)/2$ the spectrum of the harmonic oscillator Hamiltonian (42) is given by

$$E_n^{HO} = \frac{\omega}{2}\Big([n+1]_\nu + [n]_\nu \Big) = \omega(n + \nu + 1/2)\,, \qquad n = 0, 1, 2, 3, \ldots\,, \tag{45}$$

a result already know in literature [12–15,19].

The SUSY ground state, belonging to the eigenvalue $E_0 = 0$, may be calculated via (14) with $W(x) = m\omega x$ resulting in

$$\psi_0^-(x) = \sqrt{\frac{(m\omega)^{\nu+1/2}}{\Gamma(\nu + 1/2)}} \exp\left\{ -\frac{m\omega}{2} x^2 \right\}\,. \tag{46}$$

The eigenstates for the higher energy eigenvalues may then be calculated via the second relation in (38) which, for the current example, reads

$$\psi_n^-(x) = \frac{(2m\omega)^{-n/2}}{\sqrt{[1]_\nu [2]_\nu \cdots [n]_\nu}} \left(-\partial_x - \frac{\nu}{x}(1 - R) + m\omega x \right)^n \psi_0^-(x) \tag{47}$$

This may be simplified by introducing the dimensionless variable $\xi = \sqrt{m\omega}\, x$ and the abbreviation $\gamma_\nu(n) \equiv [n]_\nu ! = [1]_\nu [2]_\nu \cdots [n]_\nu$ to

$$\psi_n^-(x) = \big([n]_\nu !\big)^{-1/2} (\mathbf{A}^\star)^n \psi_0^-(x)\,, \tag{48}$$

where we have set $\mathbf{A}^\star = \left(-\partial_\xi - \frac{\nu}{\xi}(1 - R) + \xi \right) / \sqrt{2}$. This operator may be identified with the one introduced by Rosenblum, see eq. (3.4.3) in [19]. Furthermore, above relation (48) is found to be identical to relation (3.7.3) in [19] and hence we can identify the wave functions (47) with the generalized Hermite functions $\phi_n^\nu(\xi)$ introduced by Rosenblum

$$\psi_n^-(x) = \frac{(m\omega)^{\frac{[1]_\nu}{4}}}{\sqrt{\Gamma(\nu + 1/2)}} \phi_n^\nu(\sqrt{m\omega}\, x)\,, \tag{49}$$

which form a complete orthonormal set in $\mathcal{H}$.

The singular harmonic oscillator

As anticipated above, let us now consider a singular SUSY potential of the form

$$W(x) = m\omega x - \frac{\lambda}{x}\,, \qquad \lambda > 0\,. \tag{50}$$

In standard SUSY quantum mechanics this characterizes the radial harmonic oscillator problem on the positive half line where in essence λ is identified with the angular momentum quantum number ℓ. The parameter mapping $a_s = \lambda - s$ would obey the first condition in (39) but not the second. The mapping $a_s = (-1)^s \lambda$ obeys both conditions. However, with $\lambda \to -\lambda$ the condition $\lambda > 0$ is no longer fulfilled. So this will not provide us with a shape-invariant SUSY system in the current context. However, at the end of this section we will presented a modified model which has shape invariance and is closely related to the singular oscillator model in standard SUSY QM discussed in [20]. Here we note that shape-invariance is a sufficient but not a necessary condition for obtaining exact results.

Let us first investigate a possible unbroken SUSY via relation (14), which results in the two possible SUSY ground states

$$\begin{aligned} \psi_{0e}^-(x) &= \sqrt{\frac{(m\omega)^{\lambda+\nu+1/2}}{\Gamma(\lambda+\nu+1/2)}}\, |x|^\lambda \exp\{-m\omega x^2/2\}\,, \\ &\quad \text{even ground state} \\ \psi_{0o}^-(x) &= \sqrt{\frac{(m\omega)^{\lambda-\nu+1/2}}{\Gamma(\lambda-\nu+1/2)}}\, \mathrm{sgn}\,(x) |x|^{\lambda-2\nu} \exp\{-m\omega x^2/2\}\,, \\ &\quad \text{odd ground state} \end{aligned} \tag{51}$$

Obviously, for both functions to belong to the Hilbert space $\mathcal{H} = L^2(\mathbb{R}, \mathrm{d}x\, |x|^{2\nu})$ we need to require $\lambda > -\nu - \frac{1}{2}$ for the even case and $\lambda > \nu - \frac{1}{2}$ for the odd case, respectively. Here we note that with the stricter of both conditions, that is $\lambda > |\nu| - \frac{1}{2}$, the zero energy is doubly degenerate and the kernel of H_- consists of an even and an odd ground state.

The partner potentials induced by SUSY potential (50) explicitly read

$$V_\pm(x) = \frac{m}{2}\omega^2 x^2 + \frac{\lambda(\lambda \pm (2\nu R - 1))}{2mx^2} - \lambda\omega \pm \frac{\omega}{2}(2\nu R + 1)\,. \tag{52}$$

The corresponding eigenvalue problem for the partner Hamiltonians $H_\pm$ is explicitly given by

$$-\frac{1}{2m}\left[\partial_x^2+\frac{2\nu}{x}\partial_x-\frac{\nu}{x^2}(1-R)\right]\psi_E^\pm(x)+V_\pm(x)\psi_E^\pm(x)=E\psi_E^\pm(x)\,. \tag{53}$$

By construction we have $[H_\pm,R]=0$ and therefore the above eigenfunction can be classified into even and odd eigenfunction of R, i.e., $R\psi_E^\pm(x)=r^\pm(E)\psi_E^\pm(x)$, where $r^\pm(E)=1$ for even $\psi_E^\pm(x)=\psi_E^\pm(-x)$ and $r^\pm(E)=-1$ for odd $\psi_E^\pm(x)=-\psi_E^\pm(-x)$.

Let us start with the even case $r^\pm(E)=1$ and solve Eq. (53) for $\psi_E^-(x)$ via the ansatz $\psi_E^-(x)=x^{-\nu}u(x)$. This ansatz reduces the above problem to that of the radial harmonic oscillator in three dimensions with angular momentum $\ell=\lambda+\nu-1$ and energy $\varepsilon=E+\omega(\lambda+\nu+\frac{1}{2})$, which is a standard text book problem,

$$\varepsilon_n=\omega(2n+\ell+\frac{3}{2})\,,$$
$$u_n(x)=\sqrt{\frac{2m\omega\Gamma(n+1)}{\Gamma(n+\ell+1)}}\left(m\omega x^2\right)^{\frac{\ell}{2}}\mathrm{e}^{-\frac{m\omega}{2}x^2}L_n^\ell(m\omega x^2)\,. \tag{54}$$

Hence, the result for the even function is given by

$$E_n=2n\omega\,,\qquad n=0,1,2,3,\ldots$$
$$\psi_{E_n e}^-(x)=\sqrt{\frac{(m\omega)^{\lambda+\nu+1/2}\Gamma(n+1)}{\Gamma(n+\lambda+\nu+\frac{1}{2})}}\,|x|^\lambda\exp\left\{-\frac{m\omega}{2}x^2\right\}L_n^{\lambda+\nu-\frac{1}{2}}(m\omega x^2)\,, \tag{55}$$

where $L_n^\kappa(z)$ denotes the associated Laguerre polynomial of order n and the additional subindex e indicates that this is an even solution, $R\psi_{E_n e}^-(x)=\psi_{E_n e}^-(x)$. Note that $\psi_{E_n e}^-$ is properly normalized with respect to the measure (6).

The odd case $r^\pm(E)=-1$ follows the same steps as above, except that now we identify angular momentum and energy of the radial oscillator by $\ell=\lambda-\nu-1$ and $\varepsilon=E+\omega(\lambda-\nu+\frac{1}{2})$, respectively. Hence we can again take the textbook result (54). In essence one needs to replace in (55) the parameter λ by $\lambda-2\nu$. Hence, the result is

$$E_n=2n\omega\,,\qquad n=0,1,2,3,\ldots$$
$$\psi_{E_n o}^-(x)=\sqrt{\frac{(m\omega)^{\lambda-\nu+1/2}\Gamma(n+1)}{\Gamma(n+\lambda-\nu+\frac{1}{2})}}\,\mathrm{sgn}\,x|x|^{\lambda-2\nu}\exp\left\{-\frac{m\omega}{2}x^2\right\}L_n^{\lambda-\nu-\frac{1}{2}}(m\omega x^2)\,. \tag{56}$$

Recall that we have assumed $\lambda>|\nu|-\frac{1}{2}$ and therefore these odd solutions are also properly normalized on $\mathcal{H}$. The corresponding eigenstates of H_+ to the same eigenvalue E_n can be obtained with the help of the SUSY transformation (23):

$$\psi_{E_n e/o}^+(x)=\frac{1}{\sqrt{2mE_n}}\left(D_x+m\omega x-\frac{\lambda}{x}\right)\psi_{E_n o/e}^-(x)\,, \tag{57}$$

where $n=1,2,3,\ldots$.

In concluding this section let us consider a modified version of above SUSY potential by adding a non-local R-dependent term to it,

$$W(x)=m\omega x-\frac{\lambda}{x}(1-R)\,. \tag{58}$$

Obviously, the SUSY-charge operator (9), in this case, is given by

$$A=\frac{1}{\sqrt{2m}}\left(\partial_x+\frac{\nu}{x}(1-R)+W(x)\right)=\frac{1}{\sqrt{2m}}\left(\partial_x+\frac{\nu-\lambda}{x}(1-R)+m\omega x\right) \tag{59}$$

and thus is reduced to the shape-invariant harmonic oscillator as discussed in Section "The harmonic oscillator" with ν replaced by $\nu-\lambda$.

In particular the special case $\lambda=\nu$ results in the usual harmonic oscillator model on the real line. However, the Hilbert space is still equipped with the non-standard scalar product (6). With substitution $\tilde\psi(x)=|x|^\nu\psi(x)$ we can map this quantum system onto one within the usual Hilbert space $\tilde{\mathcal{H}}=L^2(\mathbb{R})$ with SUSY-charge operator given by

$$\tilde A=\frac{1}{\sqrt{2m}}\left(\partial_x+m\omega x-\frac{\nu}{x}\right)\,. \tag{60}$$

This is the shape invariant model discussed in ref. [20]. The spectrum obtained in [20] is indeed that of the WDSUSY oscillator (43).

Discussion

In the current work we have revisited the WDQM from a supersymmetric point of view. Despite the fact that several such aspects have been touched upon in the literature during the last 3 decades, the present paper considers some basic new ones. In order to study the basic features like the SUSY ground states, the SUSY transformations and a generalized shape invariance, we have limited ourselves to a class of SUSY potentials of the form $W(x)\sim x^{2k+1}$ for large x. As an example the harmonic oscillator was reconsidered from that point of view.

In addition we have considered two versions of a singular oscillator model, both of which allow for an exact solution. Whereas, the first model does not obey the shape-invariance conditions, the second version with SUSY potential (58) is shape invariant and can be mapped onto the singular model previously discussed in ref [20] within the framework of standard SUSYQM.

Another new aspect we discussed, is related to singular SUSY potentials. Whereas, for rather regular SUSY potentials only an even SUSY ground state exists, it was shown that, when adding a singular term to it, the system may also allow for an odd ground state. Hence each partner Hamiltonian has doubly degenerate energy eigenstates, an even and an odd one. As an explicit example we looked at the singular oscillator, which is exactly solvable. Here further examples, which could be looked at, are the Coulomb-like case $W(x)=c-\lambda/x$ and its non-local version $W(x)=c-\frac{\lambda}{x}(1-R)$, or SUSY potentials related to Rosen–Morse systems like $W(x)=\mu\coth x-\lambda/\sinh x$ and $W(x)=\mu\tanh x-\lambda\coth x$.

CRediT authorship contribution statement

Shi-Hai Dong: Conceptualization, Methodology, Writing – original draft, Writing – review & editing. **Won Sang Chung:** Conceptualization, Methodology, Writing – original draft, Writing – review & editing. **Georg Junker:** Conceptualization, Methodology, Writing – original draft, Writing – review & editing. **Hassan Hassanabadi:** Conceptualization, Methodology, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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